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ON THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE.*

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I.

1. We have in real quantities the identity (Euler's)†

$$1 - c_1 + c_1(1 - c_2) + c_1c_2(1 - c_3) + \dots + c_1c_2 \dots c_n(1 - c_{n+1}) \\ \equiv 1 - c_1c_2 \dots c_{n+1}.$$

Put $c_1 = a/b$, $c_{r+1} = (a + p_r)/(b + p_r)$, ($r = 1 \dots n$); multiply both sides by $b/(b - a)$ and we get

$$\frac{b}{b - a} \equiv 1 + \sum_{r=1}^n \prod_{s=1}^r \frac{a + p_{s-1}}{b + p_s} + \frac{a}{b - a} \prod_{s=1}^n \frac{a + p_s}{b + p_s}, \quad (1)$$

$p_0 = 0$. If the quantities involved be such that

$$\prod_{s=1}^{\infty} \frac{a + p_s}{b + p_s} \equiv \prod_{s=1}^{\infty} \left[1 + \frac{(a - b)/p_s}{1 + b/p_s} \right] = 0,$$

the series on the right of (1) is convergent, and has for its limit $b/(b - a)$. This infinite product vanishes when

$$(a - b) \sum_{s=1}^{\infty} \frac{1}{p_s} \frac{1}{1 + b/p_s}$$

is infinite and negative, which occurs when

$$\sum_{s=1}^{\infty} p_s^{-1} = -\infty \quad \text{and} \quad a > b,$$

$$\sum_{s=1}^{\infty} p_s^{-1} = +\infty \quad \text{and} \quad a < b,$$

a and b being positive quantities.

Evidently these conditions enable us to include among the p 's any finite number of values $q_1 \dots q_k$ we choose, a number of which may be equal, provided after q_k the quantities follow a law of formation which makes $\sum_s p_s^{-1}$ diverge to $\pm \infty$.

* Read, with some changes, before the New York Mathematical Society Oct. 7, 1893.

† Chrystal's Algebra, Vol. II, p. 393. Laurent, Traité d'Analyse, Vol. III, p. 327.

In particular, (1) converges, when $(s = 1, \dots, \infty)$

$$p_s = c + sh \quad \text{and} \quad a < b,$$

$$p_s = c - sh \quad \text{and} \quad a > b,$$

c and h being finite positive quantities.

2. Let a be a point in the plane of the complex variable z . Roll the Neumann sphere until it touches the z -plane at a , and let a' be the point infinity, the opposite end of the diameter through a . About a as a centre draw two circles with radii $r < R$, and transfer all points to the sphere. Let s and t be running points on the circumferences of these circles respectively, taken with reference to a . Let s' and t' be these same points taken with regard to a' , their moduli being $r' > R'$. Let z be any point on the surface of the zone included between the circles s and t . In virtue of (1), we have

$$\frac{1}{t - z} = \sum_0^\infty r \frac{z^{(r)}}{t^{(r+1)}}, \quad (\sum_s p_s^{-1} = +\infty) \quad (a)$$

$$\frac{1}{s - z} = \sum_0^\infty r \frac{z^{(r)}}{s^{(r+1)}}, \quad (\sum_s p_s^{-1} = -\infty) \quad (b)$$

$$\frac{1}{z - t} = \sum_0^\infty r \frac{t^{(r)}}{z^{(r+1)}}, \quad (\sum_s p_s^{-1} = -\infty) \quad (c)$$

$$\frac{1}{z - s} = \sum_0^\infty r \frac{s^{(r)}}{z^{(r+1)}}, \quad (\sum_s p_s^{-1} = +\infty) \quad (d)$$

wherein $z^{(0)} \equiv 1$, and

$$z^{(r)} \equiv z(z + p_1) \dots (z + p_{r-1}).$$

If $z = 0$ in (a), and $s = 0$ in (d), the series reduce to t^{-1} and z^{-1} respectively; while $z = 0$ in (c) and $s = 0$ in (b) cause these series to take the indeterminate form $0/0$, whose values are, however, $-t^{-1}$ and $-z^{-1}$. For example $z = 0$ in (c) gives

$$\begin{aligned} \frac{1}{-t} &= \frac{1}{z} \left[1 + \frac{t}{p_1} + \frac{t(t + p_1)}{p_1 p_2} + \frac{t(t + p_1)(t + p_2)}{p_1 p_2 p_3} + \dots \right] \\ &= \frac{1}{z} \prod_1^\infty \left[1 + \frac{t}{p_r} \right] = \frac{0}{0}. \end{aligned}$$

For this infinite product is zero,* since $\sum p_r^{-1} = -\infty$.

If the p 's vanish, (a) and (d) are convergent power series, while (b) and (c) diverge. The values represented by p , may, in general, be different in each of these four formulæ.

* Chrystal's Algebra, Vol. II, pp. 137, 393.

3. Let fz be a function which is holomorphic for all points on the zone (st), including its boundary. Then, u being a running point on the boundary of that region, we have, in virtue of Cauchy's theorem,

$$2\pi i fz = \int \frac{fudu}{u-z} = \int \frac{fzdt}{t-z} - \int \frac{fsds}{s-z}.$$

Substituting from the formulæ (a) . . . (d),

$$2\pi i fz = + \int_0^\infty \sum_r \frac{z^{(r)}}{t^{(r+1)}} ftdt - \int_0^\infty \sum_r \frac{z^{(r)}}{s^{(r+1)}} fsds, \quad (e)$$

$$2\pi i fz = + \int_0^\infty \sum_r \frac{z^{(r)}}{t^{(r+1)}} ftdt + \int_0^\infty \sum_r \frac{\delta^{(r)}}{z^{(r+1)}} fsds, \quad (f)$$

$$2\pi i fz = - \int_0^\infty \sum_r \frac{t^{(r)}}{z^{(r+1)}} ftdt - \int_0^\infty \sum_r \frac{z^{(r)}}{s^{(r+1)}} fsds, \quad (g)$$

$$2\pi i fz = - \int_0^\infty \sum_r \frac{t^{(r)}}{z^{(r+1)}} ftdt + \int_0^\infty \sum_r \frac{\delta^{(r)}}{z^{(r+1)}} fsds. \quad (h)$$

The series extending to infinity in all cases. We have assumed fz to be holomorphic all over the zone. This being the case, if any of the p 's in the last three forms occur in the zone as zeros of $z^{(r+1)}$, these points must be pre-scribed points, and the expansion holds good for all other points in the zone. If fz be holomorphic for all points in the zone with the exception of a finite number of poles $p_i \dots p_k$, and these p 's are made zeros of $z^{(r+1)}$, then the series expresses fz all over the zone.

4. Let $p_s = \pm sh$. Using Kramp's notation, we have

$$z^{r|h} \equiv z(z+h) \dots (z+\overline{r-1}h),$$

$$z^{r|-h} \equiv z(z-h) \dots (z-\overline{r-1}h).$$

(a) . . . (d) become

$$\frac{1}{t-z} = \sum_0^\infty \frac{z^{r|h}}{t^{r+1|h}}, \quad (a)'$$

$$\frac{1}{s-z} = \sum_0^\infty \frac{z^{r|-h}}{s^{r+1|-h}}, \quad (b)'$$

$$\frac{1}{z-t} = \sum_0^\infty \frac{t^{r|-h}}{z^{r+1|-h}}, \quad (c)'$$

$$\frac{1}{z-s} = \sum_0^\infty \frac{\delta^{r|h}}{z^{r+1|h}}, \quad (d)'$$

and (e) . . . (h) become

$$2\pi i f z = + \int \Sigma \frac{z^{r|h}}{t^{r+1|h}} f t dt - \int \Sigma \frac{z^{r|-h}}{s^{r+1|-h}} f s ds, \quad (e)'$$

$$2\pi i f z = + \int \Sigma \frac{z^{r|h}}{t^{r+1|h}} f t dt + \int \Sigma \frac{s^{r|h}}{z^{r+1|h}} f s ds, \quad (f)'$$

$$2\pi i f z = - \int \Sigma \frac{t^{r|-h}}{z^{r+1|-h}} f t dt - \int \Sigma \frac{z^{r|-h}}{s^{r+1|-h}} f s ds, \quad (g)'$$

$$2\pi i f z = - \int \Sigma \frac{t^{r|-h}}{z^{r+1|-h}} f t dt + \int \Sigma \frac{s^{r|h}}{z^{r+1|h}} f s ds. \quad (h)'$$

Through the point a passes a straight line whose direction is fixed by φ the argument of h . The points $h, 2h, \dots$ are equidistant points taken on this line from a progressively; the points $-h, -2h, \dots$ are taken on the line from a regressively. We regard the line aa' as divided progressively and regressively into an infinite number of equal parts with mod h . The line $n|h|e^{i\psi}$ crosses the zone in each direction going from a .

Let

$$j|h| < |s| < (j+1)|h|, \quad k|h| < |t| < (k+1)|h|.$$

Then will there be $k-j$ of the h -points in the zone on the progressive side of a and a like number on the regressive side. In (f)', $1/z^{r+1|h}$ has $k-j$ poles at the points $-sh$ ($s=j+1, \dots k$); in (g)', $1/z^{r+1|-h}$ has the same number at $+sh$; while in (h)', there are $2(k-j)$ poles in the zone at $\pm sh$. Except at these points the series (e)' . . . (h)' give the values of fz for all points in the zone.

By addition, we have

$$fz = \sum_r^{\infty} (A_r z^{r|h} + B_r z^{r|-h} + C_r z^{-r-1|h} + D_r z^{-r-1|-h}).$$

Therefore :

A function is expansible throughout its zone of holomorphism in ascending, descending, positive (progressive) and negative (regressive) factorials of the variable, except at a finite number of points on a certain diameter of the zone (the h -line).

5. In illustration of the application of (1); draw two circles about a with radii $R > \rho$, and divide the ρ -circle into n equal parts, calling the division points $p_1 \dots p_n$. Let z be any point distant $r < R$ from a .

It is evident that in the product

$$\text{mod } (z \pm p_1) \dots (z \pm p_n),$$

we are concerned with the product of the distances of z from the corners of the regular n -gon inscribed in the ρ -circle. By De Moivre's properties of the circle, we have for this product

$$\text{mod } (r^{2n} + \rho^{2n} - 2r^n \rho^n \cos n\varphi)^{\frac{1}{2}}$$

$\varphi = \angle zap_1$. Let t be a running point on the arc of the R -circle, $\theta = \angle tap_1$, we have

$$\frac{1}{t-z} = \sum_0^n \frac{z^{(r)}}{t^{(r+1)}} + \frac{z}{t-z} \prod_{s=1}^n \frac{z-p_s}{t-p_s}.$$

Now

$$\begin{aligned} \text{mod } \prod_s^n \frac{z-p_s}{t-p_s} &= \left[\frac{\rho^{2n} + r^{2n} - 2r^n \rho^n \cos n\varphi}{\rho^{2n} + R^{2n} - 2R^n \rho^n \cos n\theta} \right]^{\frac{1}{2}}, \\ &= \left[\frac{\rho}{R} \right]^n \left[\frac{1 + (r/\rho)^{2n} - 2(r/\rho)^n \cos n\varphi}{1 + (\rho/R)^{2n} - 2(\rho/R)^n \cos n\theta} \right]^{\frac{1}{2}}, \quad r < \rho \\ &= \left[\frac{r}{R} \right]^n \left[\frac{1 + (\rho/r)^{2n} - 2(\rho/r)^n \cos n\varphi}{1 + (\rho/R)^{2n} - 2(\rho/R)^n \cos n\theta} \right]^{\frac{1}{2}}, \quad r > \rho \\ &= \left[\frac{r}{R} \right]^n 2 \sin \frac{1}{2} n\varphi / [1 + (r/R)^{2n} - 2(r/R)^n \cos n\theta]^{\frac{1}{2}}, \quad r = \rho \end{aligned}$$

which vanishes when $n = \infty$, under the conditions. If fz is holomorphic all over the R -circle, we have

$$2\pi i fz = \int_0^n \sum_r \frac{z^{(r)}}{t^{(r+1)}} ftdt + \int \frac{z fz}{t(t-z)} \sum_0^n \frac{z+p_s}{t+p_s} dz.$$

Let ρ remain constant and n increase to infinity, causing the p_r 's ($r = 1, 2, \dots$) to converge to p_1 , or say, p . Then

$$2\pi i (fz - fa) = z \int \frac{ftdt}{t(t+p)} + \dots + z(z+p)^r \int \frac{ftdt}{t(t+p)^{r+1}} + \dots,$$

p being any point inside the R -circle. If $\text{mod } p = \rho$ be zero, then the series passes into Cauchy's expansion of fz .

6. In (e)' ... (h)', put $h = 0$,* we have

$$2\pi i fz = \sum_0^\infty z^r \left[\int t^{-r-1} ftdt - \int s^{-r-1} f s ds \right], \quad (\text{e})''$$

$$2\pi i fz = \sum_0^\infty \left[z^r \int t^{-r-1} ftdt + z^{-r-1} \int s^r f s ds \right], \quad (\text{f})''$$

* Note remark at the end of Art. 2.

$$2\pi i f z = - \sum_0^{\infty} \left[z^r \int s^{-r-1} f s ds + z^{-r-1} \int t^r f t dt \right], \quad (g)''$$

$$2\pi i f z = \sum_0^{\infty} z^{-r-1} \left[\int s^r f s ds - \int t^r f t dt \right]. \quad (h)''$$

The second of these is Laurent's expansion in the zone.

7. In (e)'' . . . (h)'', suppose $s = 0$, and fz holomorphic all over the t -circle, then we have in the t -circle,

$$2\pi i f z = \sum_0^{\infty} z^r \int t^{-r-1} f t dt = 2\pi i \sum z^r f^r a / r! \quad (e)'''$$

$$2\pi i f z = \sum_0^{\infty} z^r \int t^{-r-1} f t dt = 2\pi i \sum z^r f^r a / r! \quad (f)'''$$

$$2\pi i f z = - \sum_0^{\infty} z^{-r-1} \int t^r f t dt \quad (g)'''$$

$$2\pi i f z = - \sum_0^{\infty} z^{-r-1} \int t^r f t dt \quad (h)'''$$

The first two of these is Cauchy's expansion in the t -circle.

8. In (e)'' . . . (f)'', put $t = \infty$, assuming fz to be holomorphic everywhere outside of the s -circle. Then in virtue of § 1, we have for all points outside the s -circle,

$$2\pi i f z = - \sum_0^{\infty} z^r \int s^{-r-1} f s ds, \quad (e)^{iv}$$

$$2\pi i f z = \sum_0^{\infty} z^{-r-1} \int s^r f s ds, \quad (f)^{iv}$$

$$2\pi i f z = - \sum_0^{\infty} z^r \int s^{-r-1} f s ds, \quad (g)^{iv}$$

$$2\pi i f z = \sum_0^{\infty} z^{-r-1} \int s^r f s ds. \quad (h)^{iv}$$

The analogy between this set and the preceding needs no comment when we regard the circles s and t from the point a' as well as from a , remembering that $ss' = 1$.

9. We define the m th progressive difference of fz at the point z to be

$$\begin{aligned} \Delta^{m|h} &= f(z + mh) - C_{m,1} f(z + \overline{m-1}h) + \dots + (-1)^m C_{m,m} f z \\ &= (E^h - 1)^m f z, \end{aligned}$$

and the m th regressive difference at the same point to be

$$\begin{aligned}\Delta^{m|h} f z &= f z - C_{m,1} f(z-h) + \dots + (-1)^m C_{m,m} f(z-mh), \\ &= (1 - E^{1/h})^m f z.\end{aligned}$$

Suppose fz is holomorphic in a circle about a , whose radius is greater than $\text{mod}(mh)$. Then, by Cauchy's formula,

$$f a = \frac{1}{2\pi i} \int \frac{f z dz}{z-a}.$$

We have

$$\begin{aligned}\Delta^{m|h} f a &= (E^{1/h} - 1)^m f a \\ &= \frac{1}{2\pi i} \int \left[\frac{C_{m,0}}{z-a-mh} - \dots + (-1)^m \frac{C_{m,m}}{z-a} \right] f z dz, \\ &= \frac{1}{2\pi i} \int f z dz \cdot (1 - E^{1/h})^m \frac{(-1)^m}{(z-a)^{1/h}}, \\ &= \frac{1}{2\pi i} \int f z dz \cdot \Delta^{m|h} \frac{(-1)^m}{(z-a)^{1/h}}, \\ &= \frac{1}{2\pi i} \int \frac{m! h^m}{(z-a)^{m+1|h}} f z dz.\end{aligned}$$

Therefore

$$\frac{\Delta^{m|h} f a}{m! h^m} = \frac{1}{2\pi i} \int \frac{f z dz}{(z-a)^{m+1|h}}. \quad (\alpha)$$

In like manner we find

$$\frac{\Delta^{m|h} f a}{m! h^m} = \frac{1}{2\pi i} \int \frac{f z dz}{(z-a)^{m+1|h}}. \quad (\beta)$$

Let $s = 0$ in (f)', then we have

$$f z = f a + \sum_{r=1}^m \frac{z^{r|h}}{r!} \frac{\Delta^{r|h} f a}{h^r} + \sum_{m=\infty}^{\infty} z^{r|h} \int \frac{f t dt}{t^{r+1|h}}.$$

The series converges to Cauchy's with $m = \infty$, $h = 0$.